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Automated Calculation of Parametric Instabilities in Fluid Plasmas*

BERNARD ROSEN

Department of Physics and Engineering Physics, Stevens Institute of Technology, Hoboken, New Jersey

AND

HERBERT TESSER

Department of Physics, Pratt Institute, Brooklyn, New York, 11205

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A scheme for the automatic calculation of rates for parametric instabilities has been devised and has been implemented using the MACSYMA system. A survey of decay rates for modes propagating in warm fluid magnetoplasmas has begun. In this paper we discuss the method and some techniques for its implementation. The mathematical formalism used for the calculations permits a uniform treatment of these instabilities and it relies heavily on the structure of the linear modes that are coupled by the pump fields. Most of the information on these modes is generated automatically. The algorithms developed for determination of the linear modes are potentially useful for other calculations in plasma physics. An essential feature of the method is the automatic retention of only leading terms in expressions. In this regard the calculation by machine is done in humanlike fashion.

1. INTRODUCTION

We present a scheme for the automatic derivation of the analytic form of the growth rates and the frequency shifts of parametric instabilities [1–3] and its partial implementation. The method for calculating decay instability rates does not rely in an essential fashion on the breakup of fields into slow and fast modes; the decay rate for an electromagnetic mode going to two plasmons is calculated by the same procedure used for the decay of a lower hybrid wave into two other lower hybrid waves. We have implemented our algorithm for the calculation of decay instabilities occurring in a warm fluid magnetoplasma. The MACSYMA [4] system has been used for this purpose. The ultimate goal of this work is the production of a systematic survey of parametric decays and instabilities. The extension of the method presented here to Vlasov theory is in progress.

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In the next section we present the formalism used in the calculations. As will be seen the essential step for the calculation of parametric instabilities is the determination of mode-coupling coefficients. The essential difference, from the mathematical point of view, between decay instabilities and the more involved parametric processes such as an oscillating two-stream instability, is the number of modes involved. (By modes we mean the modes of the linearized system that are coupled together by pump fields.) As can be seen already from Ref. 1 the coupling coefficients involved in the oscillating two-stream case can be calculated by the same method as is used for the decay instability; it is the dispersion relation that is different in the two cases.

The calculations are done by using the leading term(s) of rather large expressions. Complete cancellations between leading terms can result in a false null answer. We calculate the next surviving germ where possible or else mark the results with an indication of possible error.

The longest part of the calculation is the determination of the mode structures of the waves participating in the instabilities. We have developed an algorithm for finding approximate analytic solutions for the sixth-order (in ω^2, ω = angular frequency) dispersion relation for the linear modes in a warm fluid magnetoplasma. The polarizations of the modes are determined by a combination of machine and human calculations. The codes developed for determining linear modes thus have potential usefulness for other problems in plasma physics.

In Section 2 we present the mathematical formalism used for the derivation of couplings and dispersion relations and in Section 3 we apply the formalism, as an illustration, to the model used by Nishikawa in Ref. 2. In Section 4 we present a step-by-step outline of the computation and in Section 5 we present details on some of the techniques used by us to solve the dispersion relation and to keep only leading terms in our results. Finally, in Section 6 we briefly summarize the cases worked out to date.

Allied work on symbolic computation of nonlinear couplings has been reported by Bers *et al.* [5]. The work presented here is complementary to theirs in that they place special emphasis on symmetry properties of the couplings and with the specific origin of various terms (box technique).

2. Mathematical Formulation

In this section we present the analytical basis of the automated calculations. As are most other calculations of these instabilities ours are limited to effects quadratic in the pump amplitude. Our discussion is based upon the use of a particular fluid model but the reader will see that the method is more generally applicable.

We start with the dynamic Maxwell equations (Faraday's law, Ampere-Maxwell law), the equations of continuity for electrons and ions, and the equations of motion for the electrons and ion fluids which are imbedded in a uniform static magnetic field. The electron fluid has a finite temperature; the ions are cold.

In this model there are 14 equations, each first order in time. There are, however,

two time-independent integrals of the motion. We eliminate the ion density using Poisson's equation and one component of the magnetic field using the fact that the divergence of this field is zero; we then have 12 equations for *field* amplitudes.

The usual assumptions that all dynamic variables have a large "pump" part (which contains the frequencies $\pm\Omega$) and a small "signal" or "disturbance part (containing frequencies ω, $\omega \pm \Omega_0$) lead us to equations that have the structure

$$\begin{pmatrix} -i(\omega + \Omega_0)\,\mathbf{F}^+ \\ -i\omega\mathbf{F}^0 \\ -i(\omega - \Omega_0)\,\mathbf{F}^- \end{pmatrix} = \begin{pmatrix} \mathbf{M}^+ & \mathbf{N}^{+0} & 0 \\ \mathbf{N}^{0+} & \mathbf{M}^0 & \mathbf{N}^{0-} \\ 0 & \mathbf{N}^{-0} & \mathbf{M}^- \end{pmatrix} \begin{pmatrix} \mathbf{F}^+ \\ \mathbf{F}^0 \\ \mathbf{F}^- \end{pmatrix}, \tag{1}$$

where the F's are multicimponent vectors of Fourier-analyzed field amplitudes of dimension m ($= 12$ in our case) and the M's and N's are $m \times m$ dimensional. In the absence of the pump the N matrices would vanish. The M's are given by the linear theory of plasma oscillations; the eigenvalues of M (multiplied by i) are the possible frequencies of the linear modes. We have truncated our equations by using the assumptions that the field amplitudes at $\omega \pm n\Omega_0$, $n > 1$ are negligible.

Using the first and last m rows of Eq. (1) we can eliminate F^+ and F^- from the central rows and thus obtain

$$\begin{gathered} (-i\omega\mathbf{1} - \mathbf{M})\,\mathbf{F}^0 = \mathbf{\Sigma}\mathbf{F}^0; \\ \mathbf{\Sigma} \equiv \mathbf{N}^{0+}(-i(\omega + \Omega_0)\,\mathbf{1} - \mathbf{M}^+)\,\mathbf{N}^{+0} + N^{0-}(-i(\omega - \Omega_0)\,\mathbf{1} - \mathbf{M}^-)^{-1}\,\mathbf{N}^{-0}. \end{gathered} \tag{2}$$

Next we introduce normalized left and right eigenvectors of the M matrices, (l_μ and r_μ, respectively, $\mu = 1, 2, \ldots, m$) and obtain our basic result for the mode amplitudes at ω:

$$-i(\omega - \Omega_\alpha{}^0)\,\mathscr{F}_\alpha{}^0 = \sum_{\beta=1}^{m} (\ell_\alpha{}^0 \cdot \mathbf{\Sigma} \cdot r_\beta{}^0)\,\mathscr{F}_\beta{}^0, \tag{3}$$

where

$$\begin{aligned} \mathbf{M}^0 r_\alpha{}^0 &= -i\Omega_\alpha{}^0 r_\alpha{}^0 \\ \ell_\alpha{}^0 \mathbf{M}^0 &= -i\Omega_\alpha{}^0 \ell_\alpha{}^0 \end{aligned}$$

and where $\mathscr{F}_\rho{}^0 \equiv \ell_\rho \cdot \mathscr{F}^0$, $\rho = \alpha, \beta$, are mode amplitudes.

In practice the calculation of the matrix $\mathbf{\Sigma}$ is relatively simple since there are usually one or two high-frequency modes, i.e., those at $\omega + \Omega_0$ and $\omega - \Omega_0$. Without loss of generality we can write

$$\begin{aligned} &\mathbf{N}^{0\pm}(-i(\omega \pm \Omega_0)\,\mathbf{1} - \mathbf{M}^\pm)^{-1}\,\mathbf{N}^{\pm 0} \\ &\quad = \sum_{\mu=1}^{m} \frac{\mathbf{N}^{0\pm} \cdot r_\mu{}^\pm \ell_\mu{}^\pm \cdot \mathbf{N}^{\pm 0}}{-i(\omega \pm \Omega_0 - \Omega_\mu{}^\pm)}, \end{aligned}$$

where $r_\mu{}^\pm$ and $l_\mu{}^\pm$ are the eigenvectors of $M^\pm$ and $\Omega_\mu{}^\pm$ are the corresponding eigenvalues. The sum on μ goes over all modes but in our applications there is only one

value of μ for which $\omega \pm \Omega_0 - \Omega_\mu^{\pm}$ is small. In other words, for the fields at the fraquencies $\omega \pm \Omega_0$ we can use the resonance approximation.

We will now apply this result to calculate rates for the various species of parametric instabilities.

2.1. *Decay Instability*

The pump wave is assumed to decay in one side band (at $\omega + \Omega_0$, say) and into the wave at ω Eq. (3) becomes

$$-i(\omega - \Omega_\alpha^0)\,\mathscr{F}_\alpha^0 = \frac{(\ell_\alpha^0 \cdot \mathbf{N}^{0+} \cdot \imath_\mu^+)(\ell_\mu^+ \cdot \mathbf{N}^{+0} \cdot \imath_\alpha^0)\,\mathscr{F}_\alpha^0}{-i(\omega + \Omega_0 - \Omega_\mu^+)}.$$

(We assume for convience that the real parts of ω and Ω_α^0 are negative.) If we define the frequency mismatch $\Delta_{\alpha\mu}$ by

$$\Delta_{\alpha\mu}^{\pm} = \Omega_\alpha^0 \pm \Omega_0 - \Omega_\mu^{\pm} \tag{4}$$

and the coupling coefficient $\nu_{\alpha,\mu}^+ \nu_{\mu,\alpha}^+$ by

$$\nu_{\alpha,\mu}^{\pm} \nu_{\mu,\alpha}^{\pm} = \ell_\alpha^0 \cdot \mathbf{N}^{0\pm} \cdot \imath_\mu^{\pm} \ell_\mu^{\pm} \cdot \mathbf{N}^{\pm 0} \cdot \imath_\alpha^0, \tag{5}$$

we obtain

$$\omega = \Omega_\alpha^0 - \tfrac{1}{2}\Delta_{\alpha\mu}^+ \pm (\tfrac{1}{4}\Delta_{\alpha\mu}^+ - \nu_{\alpha,\mu}^+ \nu_{\mu,\alpha}^+)^{1/2}. \tag{6}$$

If $\Delta_{\alpha\mu}^+$ is zero, then $(\nu_{\alpha,\mu}^+ \nu_{\mu,\alpha}^+)^{1/2}$ is the growth rate of the instability. Equation (6) is a general formula for those instability decay rates for which our assumptions are valid.

2.2. *Instabilities Involving Low-Frequency Modes*

If the frequency Ω_α^0 is low and/or the pump is sufficiently strong, then the fields at the frequency ω may not be that of one given linear mode but will be a superposition of such modes. Assume then that the modes F_α and $F_{-\alpha}$ are excited along with the side bands at Ω_μ^+ and Ω_μ^- (whose real parts are nearly equal in magnitude but of opposite sign). Equation (2) implies that

$$-i(\omega - \Omega_\alpha^0)\,\mathscr{F}_\alpha^0 = \mathscr{F}_\alpha^0(\ell_\alpha^0 \cdot \Sigma \cdot \imath_\alpha^0) + \mathscr{F}_{-\alpha}^0(\ell_\alpha^0 \cdot \Sigma \cdot \imath_{-\alpha}).$$

Using the resonance approximation for the high frequencies, one finds

$$-i(\omega - \Omega_\alpha^0)\,\mathscr{F}_\alpha^0 = \mathscr{F}_\alpha^0\left[\frac{\nu_{\alpha,\mu}^+ \nu_{\mu,\alpha}^+}{-i(\omega + \Omega_0 - \Omega_\mu^+)} + \frac{\nu_{\alpha,\mu}^- \nu_{\mu,\alpha}^-}{-i(\omega - \Omega_0 - \Omega_\mu^-)}\right]$$

$$- \mathscr{F}_{-\alpha}^0\left[\frac{\nu_{-\alpha,\mu}^+ \nu_{\mu,-\alpha}^+}{-i(\omega + \Omega_0 - \Omega_\mu^+)} + \frac{\nu_{-\alpha,\mu}^- \nu_{\mu,-\alpha}^-}{-i(\omega - \Omega_0 - \Omega_\mu^-)}\right]. \tag{7}$$

and a similar equation with α and $-\alpha$ interchanged.

The dispersion relation is given by the 2×2 determinant of Eq. (7) and, if we ignore terms quartic in the ν's, is

$$(\omega^2 - \Omega_\alpha^{02})^2 + (\omega - \Omega_{-\alpha}^0)\left[\frac{\nu_{\alpha,\mu}^+\nu_{\mu,\alpha}^+}{\omega + \Delta_{\alpha,\mu}^+ - \Omega_\alpha{}^0} + \frac{\nu_{\alpha,\mu}^-\nu_{\mu,\alpha}^-}{\omega + \Delta_{\alpha,\mu}^- - \Omega_\alpha{}^0}\right]$$

$$+ (\omega - \Omega_\alpha{}^0)\left[\frac{\nu_{-\alpha,\mu}^+\nu_{\mu,-\alpha}^+}{\omega + \Delta_{-\alpha,\mu}^+ - \Omega_{-\alpha}^0} + \frac{\nu_{-\alpha,\mu}^-\nu_{\mu,-\alpha}^-}{\omega + \Delta_{-\alpha,\mu}^- - \Omega_{-\alpha}^0}\right]. \tag{8}$$

The terms we have ignored in deriving Eq. (8) from Eq. (7) are

$$(l_\alpha \cdot \Sigma \cdot \imath_\alpha)(\ell_\alpha \cdot \Sigma \cdot \imath_\alpha) - (\ell_\alpha \cdot \Sigma \cdot \imath_\alpha)(\ell_{-\alpha} \cdot \Sigma \cdot \imath_{-\alpha})$$

and one would like to justify this neglect. This may be done by arguing that since the theory is valid only for relatively weak pump fields, the term quartic in the ν's are negligible compared to those quadratic in the ν's. Up to this point we have made two essential approximations (other than the fluid approximation), viz.:

(1) truncation of terms involving fields at frequencies $\omega \pm n\Omega_0$, $n \geqslant 2$;

(2) The pump fields are of constant amplitude and frequency; in other words pump depletion is ignored.

Neither of these assumptions is based exclusively on the small size of the pump fields and coupling coefficients. The fiirst assumption is usually based on the fact that the plasma has a small response at these higher frequencies; the second can be justified by assuming that the theory holds only while the signal fields are quite small.

There is another basis for ignoring these terms. In most couplings to low-frequency modes the coupling is either to the low-frequency density fluctuation or to $\mathbf{E} \times \mathbf{B}$ drift of electrons. If this is so and if the two modes we have designated by $\mathscr{F}_\alpha$ and $\mathscr{F}_{-\alpha}$ have the same wave vector but opposite frequencies, then the components of the eigenvectors $(\ell_{\pm\alpha}, \imath_{\pm\alpha})$ that figure significantly in the coupling are the same for both frequencies $(\pm\Omega_\alpha{}^0)$. For these cases the terms we have ignored vanish identically. In the next section, in which apply our formalism to the Nishikawa [2] model, the terms do cancel exactly.

The various N matrices, which specify the nonlinear couplings between fields, are not all independent. Each of them depend on two (wave number, frequency) variables and a pump field amplitude. which we denote here by E. We can write

$$N^{0+} = N((k, \omega), (k^+, \omega^+), E(-k_0, -\Omega_0)),$$
$$N^{0-} = N((k, \omega), (k^-, \omega^-), E(k_0, \Omega_0)),$$
$$N^{-0} = N((k^-, \omega^-), (k, \omega), E(-k_0, -\Omega_0)),$$
$$N^{+0} = N((k^+, \omega^+), (k, \omega), E(k_0, \Omega_0)),$$

where $k^\pm = k \pm k_0$, $\omega^\pm = \omega \pm \Omega_0$. Similarly we can give a general form for mode coupling coefficients $\nu_{\alpha,\mu}^{i,j}$ where

$$\nu_{\alpha,\mu}^{i,j} = \ell_\alpha{}^i \cdot \mathbf{N}^{i,j} \cdot \imath_\mu{}^j.$$

This form is

$$\nu_{\alpha,\mu}^{i,j} = \ell_\alpha{}^i(\mathbf{k}^i, \Omega_\alpha{}^i) \cdot \mathbf{N}((\mathbf{k}^i, \omega^i), (\mathbf{k}^j, \omega^j), E(\mathbf{k}^i - \mathbf{k}^j, \omega^i - \omega^j)) \cdot \imath_\mu{}^j(k^j, \Omega_\mu{}^j). \tag{9}$$

Here i and j take on the values $\pm$ and 0 while α and μ range through the number of modes. There are some obvious symmetries which can be exploited to facilitate the calculations.

3. Illustration of the Mathematical Formalism

The formulation presented in the last section was developed in order to have a uniform approach to these calculations; such uniformity is almost essential for automatic calculations. In this section we present the model of Nishikawa [2] in our notation. This model consists of two oscillators, one high frequency and one low frequency which are coupled by a pump through their displacements. The resonance approximation is not always valid for the low-frequency modes and so both low-frequency modes are needed to describe the system. Thinking of the two oscillators as representing two modes of one system we write their model in our notation, i.e., in the form of Eq. (1), with the following identifications:

$$F = \begin{pmatrix} X \\ V \end{pmatrix}, \qquad V = \frac{dX}{dt},$$

$$M^{\pm} = \begin{pmatrix} 0 & 1 \\ -\omega_H{}^2 - \Gamma^2 & 0 \end{pmatrix}, \qquad M^0 = \begin{pmatrix} 0 & 1 \\ -\omega_L{}^2 - \gamma^2 & 0 \end{pmatrix},$$

$$N^{-0} = N^{0+} = N^{+0*} = \begin{pmatrix} 0 & 0 \\ n^* & 0 \end{pmatrix}.$$

The quantity n represents the "pump" strength. The eigenvalues of the M matrices are

$$\begin{aligned} M\colon\ -i\Omega_1{}^0 &= -i\omega_L - \gamma, \\ -i\Omega_2{}^0 &= i\omega_L - \gamma; \\ M^{\pm}\colon\ -i\Omega_1{}^{\pm} &= -i\omega_H - \Gamma, \\ -i\Omega_2{}^{\pm} &= i\omega_H - \Gamma. \end{aligned}$$

If $\omega_H + \omega_L \approx \Omega_0$, then for F^-, which satisfies

$$-i(\omega - \Omega_0) F^- = M^- F^- + N^{-0} F^0,$$

the eigenvalue $\Omega_2{}^-$ holds for the nearly resonant mode, while the eigenvalue $\Omega_1{}^+$ corresponds to the nearly resonant parts of F^+. (Recall that $|\omega| \lesssim \omega_L$.)

If we denote any pair of the eigenvalues above as λ_1 and λ_2, then the corresponding left and right eigenvectors are, respectively,

$$\ell_1 = c_1(\lambda_1, -1); \qquad r_1 = c_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix},$$

$$\ell_2 = c_2(\lambda_1, -1); \qquad r_2 = c_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix},$$

$$c_1 = \frac{1}{(\lambda_2 - \lambda_1)^{1/2}}; \qquad c_2 = \frac{1}{(\lambda_1 - \lambda_2)^{1/2}}.$$

Straightforward calculation yields the Σ matrix (defined in Eq. (2)) as

$$\Sigma = \begin{pmatrix} 0 & 0 \\ -nn^* & 0 \end{pmatrix} B,$$

where

$$B = c_1^{+2} \frac{1}{-i(\omega + (\Omega_0 - \omega_H)) + \Gamma} + c_2^{-2} \frac{1}{-i(\omega - (\Omega_0 - \omega_H)) + \Gamma}.$$

Equation (7), which describes the coupling between the two F^0 modes, becomes

$$(-i(\omega - \omega_L) + \gamma)\mathscr{F}_1^0 = nn^* B(c_1^{02}\mathscr{F}_1^0 + c_1^0 c_2^0 \mathscr{F}_2^0).$$

Similarly

$$(-i(\omega - \omega_L) + \gamma)\mathscr{F}_2^0 = nn^* B(c_2^0 c_1^0 \mathscr{F}_1^0 + c_2^{02}\mathscr{F}_2^0).$$

Finally, we obtain the dispersion relation

$$\omega_L^2 - (\omega + i\gamma)^2 + nn^* B = 0$$

which agrees with that found by Nishikawa.

4. Automation of the Calculations

Our calculations consist of two disparate types. One is the generation of the coupled field equations (Eq. (1)) and the second is the calculation of mode–mode-coupling coefficients, the ν's defined in Eq. (9).

The first part starts with the appropriate equations for the plasma model (fluid, guiding-center fluid), and the electromagnetic fields and produces the M matrices (the linear part) and the N matrices (the parametric coupling portion) introduced in Eq. (1). The algorithm used for the determination of the field-coupling matrices is based upon previous work of one of the authors [6] and has been implemented in MACSYMA. Consequently, we will concern ourselves here mostly with the second half of the calculations. It should be noted that this first part need be done only once for a given model.

There is an intermediate step required between the calculation of field couplings and of mode couplings; this is the determination of the general form of left and right eigenvectors of the matrix M and also the determinant of M. The vectors are generated by use of Cramer's rule; the components of velocity and density are expressed of unspecified components of the electric field. Consequently, the polarization of the electric field of each mode involved in a given parametric process is part of the input to the program that calculates the process rate. As the reader is no doubt aware, Cramer's rule gives the solution of a set of linear equations as the ratio of two determinants; the eigenvector components we use are all multiplied by the determinant that would be in the denominator. Thus, we completely eliminate nontrivial denominators in our eigenvectors; this proves very useful in keeping only the leading terms of a sum since each term contains no factor which is itself a sum. This step also has to be carried out *once* for a given model.

As a result of these previous steps disk files containing the N matrices, the determinant of M (linear dispersion relation) and the unnormalized eigenvectors of M have been established. In order to determine a given coupling, say

$$(\ell_\alpha{}^0 \cdot \mathbf{N}^{0\pm} \cdot \iota_\mu{}^+)(\ell_\mu{}^\mp \cdot \mathbf{N}^{+0} \cdot \iota_\alpha{}^0),$$

one would "merely" have to carry out the multiplication of the N matrices by the indicated eigenvectors. In our case we use unnormalized eigenvectors and hence must multiply by the appropriate normalization factors. Finally, to put the result in more standard form one should express the pump density magnetic and velocity fields in terms of the pump electric field. In essence we do just that but with an important difference; only the terms of leading order are kept. Not only does this make the calculation more efficient, both in terms of time and computer memory required, but also gives an answer that is a comprehensible and as nearly compact as one that would be given by ordinary calculation.

The orderings used are determined from the linear theory of plasma waves [7, 8], which predicts that plasma behavior is governed by the value of various dimensionless parameters. For the warm fluid model the parameters for the plasma can be taken as the mass ratio of electrons to ions, the ratio of electron cyclotron frequency to electron plasma frequency, and the ratio of the average thermal energy per electron to the rest mass energy per electron. Furthermore, the wave numbers specified by its magnitude (in units of (speed of light)/(electron plasma frequency)) and direction relative to the magnetic field determines, upon solution of the dispersion relation, the frequencies (measured in units of the electron plasma frequency) of the linear modes. Selection of any particular mode, i.e., choice of one of these possible frequencies, then provides almost all the information necessary for the determination of the leading term(s) of each component of the eigenvectors of that mode. The additional pieces of information are the relative magnitudes of the electric field components (polarization).

The second part of our calculation, the determination of mode–mode-coupling coefficients, proceeds as follows:

(1) The parameters of the plasma as a whole are specified.

(2) For each mode, and for the pump wave, the wave number and its component parallel to the magnetic field are specified. Approximate analytic solutions of the dispersion relation are obtained by a subroutine called ROOTS which we describe in the next section. A frequency is selected and polarization information is supplied.

(3) In an optional step, under user control; information on the relative size of terms in Ampere's law and Poisson's equation is presented. These can be used to determine polarization if it has not been done in step (2).

(The next four steps are usually performed without user participation).

(4) The left and right eigenvector components of each mode are calculated to the leading order in each term. If leading terms cancel, care is taken in this step to obtain the next largest terms.

(5) The electron density, electron velocity, ion velocity, and electric and magnetic fields of the pump wave are determined, to leading order, in terms of one specified component of the electric field, say, one component perpendicular to the static magnetic field.

(6) These pump components are substituted into the coupling matrices, the N's, and again only the leading terms of each element of the coupling matrix is retained. No checking is done here to see if leading terms cancel; this point needs some attention.

(7) The various matrix–vector multiplications are now performed, and once again the leading terms are kept. In this step one checks to see if the result is as large as expected, i.e., that the leading terms have not completely cancelled out. If they have the calculation must be designated as invalid. Unlike the situation in step (4) the present case is one that involves only part of subdominant expressions and the surviving terms are possibly incomplete.

An entire calculation of the $\nu_{\alpha,\mu}\nu_{\mu,\alpha}$ involved in a parametric decay (see Eq. (5)) not including step (3) but including some factoring and compactification of the final expression and its "translation" into commonly used symbols usually takes 2–3 min of computer time. The process of factoring is not to be taken as trivial; it can often take as long as half the time of the rest of the calculation.

Our methods to obtain leading terms of expressions are discussed in the next section.

5. TECHNIQUES

In this section we will present a brief discussion of the techniques used to determine and to keep the leading terms of expressions and of the algorithm for the analytic soution of the sixth order dispersion relation.

5.1. *Leading Term Calculations*

In the following we assume an expression depends on parameters, P_i, $i = 1, 2, \ldots, n$, multiplicatively. Each parameter in turn is of order of magnitude ϵ^{m_i}, where $\epsilon \sim 1/10$.

There are three different cicrumstances in whcih we reduce expressions to their leading terms; we present our method for handling each case in turn.

(1). The quantity to be computed is a sum of terms each of which has already been computed in a previous step. Examples of such quantities are the coefficients of the dispersion relation polynomial and the components of the eigenvectors. For each term, which is proportional to $\prod_i^n P_i^{\alpha_i}$, where α_i are some integers, the quantity $\sum \alpha_i P_i$ has also been previously calculated. The α_i can be different for each term. For a given ordering, i.e., by assuming each $P_i \sim \epsilon^{m_i}$, the term considered above is seen to be of order ϵ^Q, where

$$Q \equiv \sum \alpha_i P_i \,|_{P_i = m_i} .$$

Determining the Q for each term is thus a matter of simple arithmetic and one need only accumulate the terms which possess the lowest value of Q in order to obtain the leading terms. If the sum of such terms is zero, one must then accumulate the terms corresponding to the higher value of Q and so on. We go through four tries before permitting the result to be zero.

(2). The quantities to be computed have been formed during the computations of coupling coefficients may contain denominators that are functions of the parameters, P_i. Typical of such quantities are the components of the N matrices (see Eq. (1)) *after* the substitution of various pump variables by their values in terms of one component of the pump electric field. (See steps (5) and (6) in Section 4). In this case we separately determine, for each term, the lowest power of ϵ in the numerator and the denominator and then, finally, the leading term(s) of the entire expression.

(3) The quantities in question are to be formed by multiplication of previously computed quantities whose order in ϵ are known and sums of the products of such quantities are then in turn to be formed (for example the matrix–vector and scalar products discussed in Section 4, step (7)). In this case we can determine in advance which terms are potentially of lowest order and need only calculate the sum of these. If the final result is zero, then the calculation of the quantity is deficient and the program gives a warning.

5.2. *Analytic Solution of Dispersion Relation*

In the limits of infinite ion mass the fluid plasma would have three nonzero modes of oscillation (here we count both $\pm\omega$ parts as one mode), two electromagnetic modes and one electrostatic mode. The sixth-order polynomial in ω^2 would have three zero frequency modes [9, 10]. (Naturally, in the presence of a static magnetic field, there can be hybridization of these modes).

In the case of large but finite ion to electron mass ratio there should be three low-frequency roots. We present a way in which the low-frequency roots can be separated

off and the order of the polynomial successively reduced until analytic approximations can be found for all or most of the roots.

Recall that if the dispersion relation could be factored, one would find it to have the form (no losses)

$$D(\omega) = \prod (\omega^2 - \omega_i^2) - \sum (\omega^2)^m a_m = 0,$$

where the ω_i^2 are the roots and that the coefficient of $(\omega^2)^{N-m}$ in $D(\omega)$ is the sum of the products of $-\omega_i^2$ taken m at a time. Suppose then that there is a root, ω_N^2, whose order of magnitude is smaller than that of any other root. The constant term, a_0, in $D(\omega)$ is $\prod_{i=1}^{N} (-\omega_i^2)$ but the *leading* term in a_1 would be that product of $(m-1)$ factors that did not contain $-\omega_N^2$. Consequently,

$$\omega_N^2 \simeq -a_0/a_1 .$$

In other words the root occurs when the last two terms in $D(\omega)$ nearly balance. By applying this reasoning to the remaining *small* roots we can determine them as being approximately equal to the ratio of adjacent coefficients in $D(\omega)$. There is a complication caused by the presence of two small roots of equal magnitude. In this case one sees that the roots occur when three adjacent terms in $D(\omega)$ nearly balance; the two roots are obtained by solving a quadratic equation. This is easily generalized to the case of three or four such roots.

If the wavenumber is large (in our units), then some of the roots vary as k^2 and are consequently large. In this case we apply the same technique but start with the coefficients of the highest powers of ω^2. In the case of one root much larger than the others, say, ω_1^2, $a_{N-1} \approx -\omega_1^2$ and since $a_N = 1$, $\omega_1^2 \approx -a_{N-1}/a_N$. Another way to come to the same result is to consider $1/\omega^{2N} D(\omega)$ as a polynomial in $1/\omega^2$. In this variant of the preceding method we separate out the high-frequency waves.

6. Summary

We have presented a scheme for the automatic computation of mode–mode couplings in parametric processes to lowest surviving order and have presented details of its implementation.

To date we have been concerned principally with the correctness of the calculations and the efficiency of the computation. Our test calculations, which verify results obtained previously [11] have been (no losses included)

(1) Raman scattering (No static magnetic field),

(2) Brillouin scattering (no static magnetic field),

(3) electromagnetic decay into two plasmons (no static magnetic field),

(4) lower hybrid decay into lower hybrid + ion acoustic wave,

(5) lower hybrid decay into two lower hybrids waves,

(6) magnetic pumping producing ion acoustic wave and fast Alfven wave.

In this last case the pump fields ($B_{\text{pump}} \parallel B_{\text{static}}$) were specified by the user rather than determined as a solution of the field equations; this is an option available in our system.

In all these cases we did not include losses explicitly; losses can be included in an ad hoc fashion by giving the eigenfrequencies ($\Omega^{\pm}$, Ω^{0}) imaginary parts. On the other hand we do not use the dipole approximation explicitly; the method of calculation may cause terms proportional to k_0 to be dropped if the pump has a long wavelength.

In the future we intend to extend the methods presented here to include losses, inhomogeneities, and kinetic effects.

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